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ON TRANSIENT CREEP BOUNDS AND APPROXIMATE SOLUTIONS FOR THE TORSION OF THIN STRIPS†

WARREN S. EDELSTEIN
Department of Mathematics, Illinois Institute of Technology, Chicago, IL 60616, U.S.A.

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Abstract—Integral equations are derived which govern transient primary and secondary creep in thin rectangular strips subject to torsion. Formal similarity between these equations and others arising in previous work are exploited to obtain bounds, monotonicity and convexity of the stress profile as well as uniform approximations.

1. INTRODUCTION

In [1–4] respectively, the problems of transient creep, both primary and secondary and including the effect of elastic strains, were studied for cylindrical and spherical pressure vessels, beams subject to pure bending and the torsion of circular cylinders. In all cases, the system of governing equations was reduced to a single nonlinear integral equation which, despite the diversity of the original problems, has the same basic form. For this reason, the same methods could be applied in all of these cases to obtain *a priori* bounds, the shape of stress profiles and approximate solutions.

The present work extends this treatment to the problem of transient creep in a thin rectangular strip undergoing torsion. The resulting integral equation takes the form (2.22) for secondary creep and (2.26) for primary creep. The various bounds, monotonicity properties, etc. then follow from a simple comparison of those equations with corresponding equation in [3, 4] and without the need for any further analysis. It is important to note that the bounds herein derived for the solution $\sigma = \sigma_{zx}$ of (2.22) or (2.26) imply upper and lower bounds for all the other non-zero quantities of interest, strain, displacement and specific angle of twist, at all times and at all points in the body.

These results would appear to be of interest in their own right, since it is in thin or slender members that creep effects are of the most concern to designers. Furthermore, the demonstration of yet another physical situation to which the methods of [2–4] apply suggests the existence of a whole class of such problems and contributes to the justification of further study of integral equations of the form (2.22) and (2.26) and their generalizations. Although most transient creep problems are much too complicated to be reduced to such equations, it is hoped that these restricted results will serve to suggest or encourage the development of more widely applicable bounding techniques. The engineering applications of such bounds have already been described in [4], Section 4.

In Section 2, the governing equations for St.-Venant torsion are set forth, the thin strip assumption (2.8) is applied and (2.22), (2.26) are derived. It seems to be more appropriate in the transient creep problem to work directly with stresses rather than with a stress function, as was done for the steady state case [5]. In Section 3, correspondences between (2.22) and 2.26) and certain integral equations in [3–4] are exploited to obtain the monotonicity (3.3) and convexity (3.7) of the stress profile, *a priori* bounds (3.5) and (3.11) and convergence of approximate solutions under various assumptions on the creep law (1.3b).

As in previous work, the infinitesimal strains ϵ_{ij} are assumed to have the form

$$\epsilon_{ij} = \epsilon_{ij}^{(e)} + \epsilon_{ij}^{(c)}, \tag{1.1}$$

where $\epsilon_{ij}^{(e)}$ and $\epsilon_{ij}^{(c)}$ denote elastic strains and creep strains respectively. These are related to the stresses σ_{ij} by the equations†

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†Subscripts have the range 1, 2, 3, δ_{ij} stands for the Kronecker delta, and summation over repeated indices is implied. We shall also use a superposed dot to denote differentiation with respect to time. Points in three-space are designated either (x_1, x_2, x_3) or (x, y, z).

$$\epsilon_{ij}^{(e)} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}], \tag{1.2}$$

$$\epsilon_{ij}^{(c)}|_{t=0} = 0 \tag{1.3a}$$

$$\dot{\epsilon}_{ij}^{(c)} = \frac{F(\sigma_e)}{[\epsilon_e^{(c)}]^m} \cdot s_{ij}, \quad t > 0. \tag{1.3b}$$

Here s_{ij} stands for the deviatoric components of the stress, σ_e is the effective stress and $\epsilon_e^{(c)}$ is the effective creep strain. They are defined by the formulas

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3} \cdot \sigma_{kk}. \tag{1.4}$$

$$\sigma_e = \sqrt{\left(\tfrac{3}{2} s_{ij}s_{ij}\right)}, \quad \epsilon_e^{(c)} = \sqrt{\left(\tfrac{2}{3} \epsilon_{ij}^{(c)} \epsilon_{ij}^{(c)}\right)}. \tag{1.5}$$

For $m = 0$, (1.3b) gives a generalized secondary creep law; for $m > 0$, (1.3b) generalizes the primary creep law

$$\dot{\epsilon}_{ij}^{(c)} = \frac{3K\sigma_e^{n-1}}{2[\epsilon_e^{(c)}]^m} \cdot s_{ij} \tag{1.6}$$

of Odqvist and Hult[6].

The infinitesimal strain-displacement relations are given by

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.7}$$

and the quasi-static stress equations of equilibrium by

$$\sigma_{ij,j} = 0. \tag{1.8}$$

It is assumed that

$$E > 0, \quad -1 < \nu \le \tfrac{1}{2}, \quad m \ge 0. \tag{1.9}$$

2. THE THIN STRIP EQUATION

We consider a rectangular strip whose cross-section (Fig. 1) lies in the x, y plane whose sides are traction-free and whose ends are subjected to the relaxed St-Venant end conditions

$$\int \sigma_{zx}\, dA = \int \sigma_{zy}\, dA = \int \sigma_{zz}\, dA = 0 \tag{2.1}$$

$$\int \sigma_{zz} y\, dA = \int \sigma_{zz} x\, dA = 0, \tag{2.2}$$

$$\int (\sigma_{zy}x - \sigma_{zx}y)\, dA = M(t). \tag{2.3}$$

Following the usual procedure for small strain torsion problems, we assume that the displacements u_x, u_y, u_z have the form†

$$u_x = -\alpha(t)zy, \quad u_y = \alpha(t)zx, \quad u_z = \alpha(t)\psi(x, y). \tag{2.4}$$

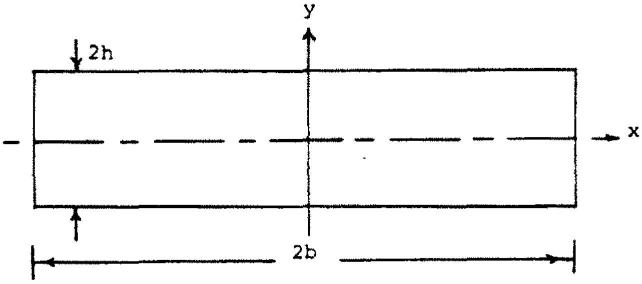

Fig. 1.

†That the warping function ψ is assumed independent of time follows the example of the treatment of the corresponding steady-state problem [5].

Therefore,

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \epsilon_{xy} = 0 \tag{2.5}$$

identically in the strip for all time. Due to the constitutive relations, it is consistent with (2.5) to set

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0. \tag{2.6}$$

It then follows from (1.8) that

$$\sigma_{zx} = \sigma_{zx}(x, y, t), \sigma_{zy} = \sigma_{zy}(x, y, t), \sigma_{zx,x} + \sigma_{zy,y} = 0. \tag{2.7}$$

The basic thin strip assumption is that†

$$\sigma_{zy} = 0 \quad (-b < x < b, \ -h < y < h, t > 0). \tag{2.8}$$

Applying this to (2.7), we get

$$\sigma_{zx} = \sigma_{zx}(y, t) \equiv \sigma(y, t). \tag{2.9}$$

If we then require σ to be an odd function of y, (2.1) and (2.2) are met. In this case, we need only consider σ for $0 < y < h$ subject to the boundary condition

$$\sigma(0, t) = 0. \tag{2.10}$$

It also follows that $\epsilon_{zx}^{(e)}$ and $\epsilon_{zx}^{(c)}$ are, respectively, the only non-zero elastic and creep strains and that

$$\sigma_e = \left(\tfrac{3}{2}\right)^{1/2} |\sigma|, \quad \epsilon_e^{(c)} = \left(\tfrac{2}{3}\right)^{1/2} |\epsilon_{zx}^{(c)}|. \tag{2.11}$$

It follows from

$$\epsilon_{zy} = 0, \tag{2.12}$$

together with (1.7) and (2.4), that

$$0 = \frac{\alpha}{2}\left(x + \frac{\partial \psi}{\partial y}\right). \tag{2.13}$$

Since, in general, $\alpha \neq 0$, we must have

$$\psi = -xy + f(x). \tag{2.14}$$

By (1.7), (2.4) and (2.14),

$$\epsilon_{zx} = \frac{\alpha}{2}(-2y + f'(x)).$$

Since, from the constitutive relations and (2.9), we may infer that ϵ_{zx} is independent of x, it follows that $f'(x)$ is constant. This constant must be zero due to (2.10). Therefore,

$$\psi = -xy + c, \quad \epsilon_{zx} = -\alpha y. \tag{2.15}$$

We are now in a position to obtain the basic creep equation. By (2.15), (1.1) and (1.2),

$$-\alpha y = \frac{(1+\nu)}{E}\sigma(y, t) + \epsilon_{zx}^{(c)}(y, t). \tag{2.16}$$

In order to eliminate α from (2.16), we first notice that (2.3) now has the form

$$-4b \int_0^h \sigma(y, t) y \, dy = M(t). \tag{2.17}$$

Thus, if we multiply both sides of (2.16) by $-4by$ and integrate with respect to y from 0 to h,

†This assumption, together with (2.6), assures that the flat sides of the strip are traction-free. In return, we must relax the boundary condition that $\sigma_{xz} = 0$ at $x = \pm b$.

we obtain the following representation for the specific angle of twist, α:

$$\alpha(t)=\frac{(1+\nu)}{4bEI}M(t)-\frac{1}{I}\int_o^h \epsilon_{zx}^{(c)}(y,t)\,y\,\mathrm{d}y,\quad I=\int_o^h y^2\,\mathrm{d}y. \tag{2.18}$$

Substituting this expression back into (2.16), we get

$$\frac{\sigma(y,t)}{E}=-\frac{M(t)y}{4bEI}+\frac{y}{I}\int_o^h\frac{\epsilon_{zx}^{(c)}(\xi,t)}{1+\nu}\,\xi\,\mathrm{d}\xi-\frac{\epsilon_{zx}^{(c)}(y,t)}{1+\nu}. \tag{2.19}$$

The easiest case to consider is secondary creep, which is obtained by setting $m=0$ in (1.3b). Then,

$$(1+\nu)^{-1}\epsilon_{zx}^{(c)}(y,t)=\int_0^t G(\sigma)\,\mathrm{d}\tau, \tag{2.20}$$

where

$$G(\sigma)\equiv(1+\nu)^{-1}F\left(\left[\tfrac{3}{2}\right]^{1/2}|\sigma|\right)\sigma. \tag{2.21}$$

The integral equation governing secondary creep in a thin strip now becomes

$$\frac{\sigma(y,t)}{E}=-\frac{M(t)y}{4bEI}+\frac{y}{I}\int_o^t\int_o^h G[\sigma(\xi,t)]\,\xi\,\mathrm{d}\xi\,\mathrm{d}\tau-\int_o^t G[\sigma(y,t)]\,\mathrm{d}\tau. \tag{2.22}$$

Primary creep ($M>0$) is governed by a system of two equations consisting of (2.19) together with

$$\dot{\epsilon}_{zx}^{(c)}=F\left(\left[\tfrac{3}{2}\right]^{1/2}|\sigma|\right)\sigma/\left[\tfrac{2}{3}\right]^{m/2}|\epsilon_{zx}^{(c)}|^m,\quad \epsilon_{zx}^{(c)}|_{t=0}=0. \tag{2.23}$$

If either m is an even integer or $\epsilon_{zx}^{(c)}>0$, (2.23) can be integrated to give

$$(1+\nu)^{-1}\epsilon_{zx}^{(c)}=\left[\int_o^t G(\sigma)\,\mathrm{d}\tau\right]^{1/(m+1)}, \tag{2.24}$$

where

$$G(\sigma)\equiv\frac{(m+1)}{(1+\nu)^{m+1}}\left[\frac{3}{2}\right]^{m/2}F\left(\left[\tfrac{3}{2}\right]^{1/2}|\sigma|\right)\sigma. \tag{2.25}$$

In this case, (2.19) becomes

$$\frac{\sigma(y,t)}{E}=\frac{-M(t)y}{4bEI}+\frac{y}{I}\int_o^h\left[\int_o^t G(\sigma)\,\mathrm{d}\tau\right]^{1/(m+1)}\eta\,\mathrm{d}\eta-\left[\int_o^t G(\sigma)\,\mathrm{d}\tau\right]^{1/(m+1)}. \tag{2.26}$$

From now on, σ will always be assumed positive on $(0,c]\times[0,\infty)$.

Notice that, mathematically, (2.26) is a special case of the integral equation (2.24) of [3] for the St.-Venant pure bending problem

$$\frac{\sigma(x,t)}{E}=-\frac{M(t)x}{EI}+\frac{x}{I}\int_o^c\left[\int_o^t G(\sigma)\,\mathrm{d}\tau\right]^{1/(m+1)}k(\xi)\,\mathrm{d}\xi-\left[\int_0^t G(\sigma)\,\mathrm{d}\tau\right]^{1/(m+1)},\qquad \text{(2.24) of [3].}$$

$$I=\int_o^c xk(x)\,\mathrm{d}x,$$

in which σ is a torsion and M is a bending moment.

In [3], it was formally shown that, for the power law (1.6), solutions σ of (2.24) have the large time representation

$$\sigma(x,\infty)=\frac{-M(\infty)x^{(m+1)/n}}{\int_o^c \xi^{(m+1)/n}k(\xi)\,\mathrm{d}\xi},\qquad \text{(2.29) of [3]}$$

provided M tends to a limit $M(\infty)$, and $\dot{M}(\infty)$ equals zero.

By comparison of (2.26) of the present paper with (2.24) of [3], we see that the thin strip shear stress $\sigma(y, t)$ should tend to

$$\sigma(y, \infty) = -\frac{M(\infty)[(m+1)/n+2]y^{(m+1)/n}}{4bh^{(m+1)/n+2}}. \tag{2.27}$$

From this result, one readily obtains the limiting creep rate, provided that the above assumptions on the M hold. In fact, from (2.18) and (2.24) it follows that

$$\alpha(t) = \frac{(1+\nu)}{4bEI}M(t) - \frac{(1+\nu)}{I}\int_o^h \left[\int_o^t G(\sigma)\,d\tau\right]^{1/m+1} y\,dy.$$

Thus, formally, $\dot{\alpha}(\infty) = 0$ for primary creep, while

$$\dot{\alpha}(\infty) = -\frac{(1+\nu)}{I}\int_o^h G[\sigma(y,\infty)]\,y\,dy \tag{2.28}$$

for secondary creep. Assuming the power law (1.6), this specializes to

$$\dot{\alpha}(\infty) = \operatorname{sgn} M(\infty)\left(\tfrac{3}{2}\right)^{(n+1)/2}\frac{K}{I}\left[\frac{|M(\infty)|([m+1]/n+2)}{4bh^{1/n+2}}\right]^n \frac{h^3}{m+3}. \tag{2.29}$$

It is easy to check that (2.27) and (2.29) agree with the secondary creep steady-state solution given in Section 49 of [5].

For general creep law (13b), the results of [3] imply that

$$\sigma(y,\infty) = G^{-1}(A^{m+1}y^{m+1}), \tag{2.30}$$

where G^{-1} denotes the functional inverse of G and A is determined by the equation

$$\int_o^h G^{-1}(A^{m+1}y^{m+1})\,y\,dy = -\frac{M(\infty)}{4b}. \tag{2.31}$$

3. BOUNDS AND APPROXIMATE SOLUTIONS

The case of primary creep will be dealt with first. In order to exploit the results of [3] for the pure bending problem, we restrict ourselves to positive solutions of (2.26) and assume that M is continuously differentiable on $[0, \infty)$. Also (See (2.6) of [3]),

$$M(0) < 0, \dot{M} \le 0 \quad (0 < t < \infty). \tag{3.1}$$

The required constitutive assumptions are ((3.3) of [3])

$$G(\sigma) > 0, G'(\sigma) > 0 \quad (0 < \sigma < \infty). \tag{3.2}$$

It then follows from (3.5) of [3] that

$$\frac{\partial\sigma}{\partial y} \ge 0 \quad (0 < y \le h, t \ge 0). \tag{3.3}$$

For the main primary creep bound to hold, we must assume the power law (1.6) subject to the restrictions

$$K > 0, n \ge m+1. \tag{3.4}$$

Then (3.9) of [3] implies that

$$0 \le \sigma(y,t) \le -\frac{hM(t)}{4bI} \quad (0 \le y \le h, t \ge 0). \tag{3.5}$$

As in [3], this result may be used together with (2.26) to obtain other bounds which tend to the exact solution as $t \to 0$.

In order to exploit previously developed secondary creep results, one must add to (3.1), (3.2) the assumptions

$$m = 0, G(0) = 0, G'' > 0. \tag{3.6}$$

It then follows from (3.20) of [3] that

$$\frac{\partial^2\sigma}{\partial y^2} \leqq 0 \quad (0 \le y \le h, t > 0). \tag{3.7}$$

More importantly, due to (3.6), one can apply to the thin strip problem the bounding and approximation results of [4] pertaining to solutions $s(r,t)$ of the integral equation

$$s(r,t)=\frac{r^l}{I}\left(N(t)+\int_0^t\int_a^b H(s)q(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\right)-\int_0^t H(s)\,\mathrm{d}\tau \quad (a\le r\le b,\ t\ge 0), \tag{3.8}$$

$$I=\int_a^b \xi^l q(\xi)\,\mathrm{d}\xi. \tag{3.9}$$

Clearly, the secondary creep thin strip eqn (2.22) constitutes that special case of (3.8) for which

$$l=1,\ q(\xi)=\xi,\ N=-\frac{M}{4b},\ H=EG,\ a=0,\ b=h. \tag{3.10}$$

With these correspondences, we obtain from [4] a monotone nonincreasing sequence $\{s_k\}$ and a monotone nondecreasing sequence $\{\sigma_k\}$ of functions on $[0,h]\times[0,\infty)$ such that, for $k=0,1,2,\ldots,$

$$\sigma_k(y,t)\le\sigma(y,t)\le s_k(y,t) \quad (0\le y<h,\ t\ge 0) \tag{3.11}$$

and

$$\lim_{k\to\infty}\sigma_k(y,t)=\lim_{k\to\infty}s_k(y,t)=\sigma(y,t)$$

uniformly in any rectangle $[0,h]\times[0,T]$. Here $\sigma(y,t)$ is the solution of (2.22) and $\{s_k\}$, is defined recursively as follows:

$$s_o(r,t)=G^{-1}\left[\frac{r}{h}G\left(\frac{-hM(t)}{4bI}\right)\right], \tag{3.12}$$

$$s_{k+1}+E\int_o^t G'(s_k)s_{k+1}\,\mathrm{d}\tau=\frac{y}{I}\left(\frac{-M}{4b}+\int_0^t\int_0^h EG(s_k)\,\xi\,\mathrm{d}\xi\,\mathrm{d}\tau\right)+E\int_0^t[G'(s_k)s_k-G(s_k)]\,\mathrm{d}\tau. \tag{3.13}$$

The above follow, respectively, from (3.8) and (3.10) of [4]. To obtain the iteration scheme for σ_k, one uses (3.20) of [4]. However, instead of the definition (3.19) of σ_o, which was derived for the case $a>o$, one sets $\sigma_o=0$. Notice that s_1, σ_1 can be obtained in closed form.

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